A Theory of Bilateral Oligopoly*

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May 3, 2005

Abstract

In hoirzontal mergers, concentration is often measured with the Hirschmann-Herfindahl Index (HHI). This index yields the price-cost margins in Cournot competition. In many modern merger cases, both buyers and sellers have market power, and indeed, the buyers and sallers may be the same set of firms. In such cases, the HHI is inapplicable. We develop an alternative theory that has similar data requirements as the HHI, applies to intermediate good industries with arbitrary numbers of firms on both sides, and specializes to the HHI when buyers have no market power. The more inelastic is the downstream demand, the more captive production and consumption (not traded in the intermediate market) affects price-cost margins. The analysis is applied to the merger of the California gasoline refining and retail assets of Exxon and Mobil.

* **Acknowledgement:** We thank Jeremy Bulow and Paul Klemperer for useful remarks and for encouraging us to explore downstream concentration.

1 Introduction

The seven largest refiners of gasoline in California account for over 95% of the production of CARB (California Air Resources Board certified) gasoline sold in the state. The seven largest brands of gasoline also accounts for over 97% of retail sales of gasoline. Thus, the wholesale gasoline market in California is composed of a number of large sellers and large buyers who compete against each other in the downstream retail market. What will be the effect of a merger of vertically integrated firms on the wholesale and retail markets? This question has relevance with the mergers of Chevron and Texaco, Conoco and Phillips, Exxon and Mobil, and BP/Amoco and Arco, all of which have been completed in the past decade.

When monopsony or oligopsony faces an oligopoly, most analysts consider that the need for protecting the buyers from the exercise of market power is mitigated by the market power of the buyers and vice versa. Thus, even when the buyers and sellers are separate firms, an analysis based on dispersed buyers or dispersed sellers is likely to err. How should antitrust authorities account for the power of buyers and sellers in a bilateral oligopoly market in evaluating the competitiveness of the market? Merging a net buyer with a net seller produces a more balanced firm, bringing what was formerly traded in the intermediate good market inside the firm. Will this vertical integration reduce the exercise of market power and produce a more competitive upstream market? Or will the vertically integrated firm restrict supply to other non-integrated buyers, particularly if they are rivals in the downstream market?

There is a voluminous literature that address these questions. Most of the literature considers situations in which one or two sellers supply one or two buyers and models their interactions as a bargaining game.[1] Sellers negotiate secret contracts with buyers specifying a quantity to be purchased and transfers to be paid by the buyer. These bilateral bargaining

[1] See Rey and Tirole [26] for a survey of this literature. Several of the main papers in this literature are Hart and Tirole [12], McAfee and Schwartz [20], O'Brien and Shaffer [24], Segal [29], and de Fontenay and Gans [5].

models do not describe intermediate good markets like the wholesale gasoline market in California. The market consists of more than two sellers and two buyers, and trades occur at a fixed, and observable, price. Other papers study vertical mergers by assigning the market power either to buyers or to sellers, but not both.[2] These models are excellent for assessing some economic questions, including the incentive to raise rival's cost, the effects of contact in several markets, or the consequences of refusals-to-deal. But, they do not address the implications of bilateral market power that we wish to study in this paper.

Traditional antitrust analysis presumes dispersed buyers. Given such an environment, the Cournot model (quantity competition) suggests that the Hirschman-Herfindahl Index (HHI, which is the sum of the squared market shares of the firms) is proportional to the price-cost margin, which is the proportion of the price that is a markup over marginal cost. Specificially, the HHI divided by the elasticity of demand equals the price-cost margin. The HHI is zero for perfect competition and one for monopoly. The HHI has the major advantage of simplicity and low data requirements. In spite of well-publicized flaws, the HHI continues to be the workhorse of concentration analysis and is used by both the US Department of Justice and the Federal Trade Commission. The HHI is inapplicable, however, to markets where the buyers are concentrated, particularly if they compete in a downstream market.

Our objective in this paper is to offer an alternative to the HHI analysis with (i) similar informational requirements, (ii) the Cournot model as a special case, and (iii) an underlying game as plausible as the Cournot model. The model we offer suffers from the same flaws as the Cournot model. It is highly stylized, and a static model. It uses a "black box" pricing mechanism motivated by the Cournot analysis. Moreover, our model will suffer from the same flaws as the Cournot model in its application to antitrust analysis. Elasticities are treated as constants when they are not, and the relevant elasticities are taken as known. However, the analysis can be applied to markets with arbitrary numbers of sellers and buyers, who individually have the power to influence price, and buyers who may compete

[2] See for example, Hart and Tirole [12], Ordover, Saloner, and Salop [25], Salinger [27], Salop and Scheffman [28], Bernheim and Whinston [2]. An alternative to assigning the market power to one side of the market is Salinger's sequential model.

against each other in a downstream market. The analysis is simple to apply, and permits the calculation of antitrust effects in a practical way.

Our approach is based on the Klemperer and Meyer [14] market game. In their model, sellers submit supply functions and behave strategically, buyers are passive, reporting their true demand curves, and price is set to clear the market. We allow the buyers to behave strategically in submitting their demand functions, and apply the same concept of equilibrium as Klemperer and Meyer. As is well known, supply function models have multiple equilibria. Klemperer and Meyer [14] reduce the multiplicity by introducing stochastic demand, and they show that, if the support is unbounded, then the equilibrium is unique and the equilibrium supply schedule is linear. Green and Newberry [11], Green [9][10], and Akgun [1] obtain uniqueness by restricting the supply schedules to be linear. Our approach is similar but we do not require linearity. In our model, sellers can select from a one-parameter family of schedules indexed by production capacity, and buyers can select from a one-parameter family of schedules indexed by consumption capacity. Thus, sellers can exaggerate their costs by reporting a capacity that is less than it in fact is, and buyers can understate their willingness to pay by reporting a capacity that is greater than it in fact is.

Our model describes the electricity balancing market in Texas[3] Generating firms submit supply schedules and utilities submit demands to an independent system operator (ISO) who chooses price to equate reported supply to reported demand and tells each generating firm to supply the amount that they said they are willing to supply at that price. More generally, our model should be interpreted as a reduced form model of a dynamic game. Our results have important implications for the vast empirical literature on estimating markups in homogenous good industries. In estimating the first-order conditions obtained from the Cournot model, empirical industrial organization economists often include a conduct parameter that they intepret as the firms' conjecture on how aggregate supply will change with an increase in their output.[4] This approach has been criticized by theorists on the

[3] See Hortescu and Puller [13].

[4] See Bresnahan [4] for a description of the methodology and a survey of a number of empirical studies.

grounds that firms cannot expect rivals to react to their output choices in a static model. Furthermore, if reactions are important, then they need to be made explicit by specifying the timing of decisions so that the firms' expectations about rivals' responses are not introduced into the model in an ad hoc way. Our model is static, but the firms' expectations about their rivals' responses to their choice of strategy are determined as part of the equilibrium of the model. However, the expectation is endogenous, and cannot be treated as a structural parameter. Our analysis provides a potential explanation for why the empirical literature finds that markets are usually much more competitive than the Cournot models would predict.

In a traditional assessment of concentration according to the U.S. Department of Justice Merger guidelines, the firms' market shares are intended, where possible, to be shares of capacity. This is surprising in light of the fact that the Cournot model does not suggest the use of capacity shares in the HHI, but rather the share of sales in quantity units (not revenue). Like the Cournot model, the present study suggests using the sales data, rather than the capacity data, as the measure of market share. Capacity plays a role in our theory, and indeed a potential test of the theory is to check that actual capacities, where observed, are close to the capacities consistent with the theory.

The merger guidelines assess the effect of the merger by summing the market shares of the merging parties.[5] Such a procedure provides a useful approximation but is inconsistent with the theory (either Cournot or our theory), since the theory suggests that, if the merging parties' shares don't change, then the prices are unlikely to change as well. We advocate a more computationally-intensive approach, which involves estimating the capacities of the merging parties from the pre-merger market share data. Given those capacities, we then estimate the effect of the merger on the industry, taking into account the incentive of the merged firm to restrict output (or demand, in the case of buyers).

The next section presents a market game. The second section derives the equilibrium

More recent studies include Genesove and Mullin [8] and Clay and Troesken [6]

[5] Farrell and Shapiro [7] and McAfee and Williams [19] independently criticize the Cournot model while using a Cournot model to address the issue.

price/cost margins and the value/price margins, which is the equivalent for buyers, for vertically separated markets and for vertically integrated markets. The third section analyzes the constant elasticity case and mergers. The fourth section investigates the effects of downstream market power on the analysis of section 3. The fifth section applies the analysis to the merger of the California assets of Exxon and Mobil, to illustrate the plausibility and applicability of the theory. The sixth section concludes.

2 The Model

We begin with a standard model of a market for a homogenous intermediate good Q. There are n firms, indexed by i from 1 to n. Each seller i produces output x_i using a constant returns to scale production function with fixed capacity γ_i. Thus, seller i's production costs takes the form

$$C(x_i, \gamma_i) = \gamma_i c\left(\frac{x_i}{\gamma_i}\right), \tag{1}$$

where $c(\cdot)$ is convex and strictly increasing.[6] Each buyer j consumes intermediate output q_j and values that consumption according to a function $V(q_j, k_j)$ where k_j is buyer j's capacity for processing the intermediate output. We assume that V is homogenous of degree one so that it can expressed as

$$V(q_j, k_j) = k_j v\left(\frac{q_j}{k_j}\right), \tag{2}$$

where $v(\cdot)$ is concave and strictly increasing.[7] A firm may be both a seller and a buyer, that is, it may produce the intermediate good and also consume it. Such firms are called vertically integrated, although they may be net sellers or net buyers. We will refer to a market with no vertically integrated firms as a vertically separated market and a market with one or more vertically integrated firms as vertically integrated market.

The assumption of constant returns to scale in production and consumption plays a pivotal role in our analysis. It is common to impose this restriction on production functions but constant returns to scale of the valuation function of buyers deserves more discussion.

[6] In addition, we assume that $c'(z) \longrightarrow \infty$ as $z \longrightarrow \infty$.

[7] In addition, we assume $v'(z) \longrightarrow 0$ as $z \longrightarrow \infty$.

We have in mind two kinds of intermediate markets, depending upon whether buyers are manufacturing firms or retail firms. If buyer j is a manufacturing firm, it combines the intermediate input q_j with capacity k_j to produce a good y_j using a constant returns to scale technology $F(q_j, k_j)$ and sells the output in a competitive market at price r per unit. In this case, the firm's revenues are given by

$$V(q_j, k_j) = rF(q_i, k_j) = rk_j f\left(\frac{q_j}{k_j}\right)$$

A manufacturing firm that has twice the capacity of another firm can produce twice as much at the same average productivity. If buyer j is a retail firm, it purchases q_j in the wholesale market to sell to final consumers at a competitive retail price r. In this case, $y_j = q_j$, and firm j's gross profits are

$$V(q_j, k_j) = rq_j - k_j w\left(\frac{q_j}{k_j}\right) = k_j\left[r\left(\frac{q_j}{k_j}\right) - w\left(\frac{q_j}{k_j}\right)\right],$$

here w represents unit selling costs. A retailer with twice as much selling capacity (e.g., number of stores) can sell twice as much at the same unit cost.

In the Cournot model, buyers report their true willingness-to-pay schedules, sellers submit quantities, and the market chooses price to equate reported supply to demand. In equilibrium, price is equal to each buyer's marginal willingness-to-pay but exceeds each seller's marginal cost of supply. In the standard oligopsony model, sellers submit their true marginal cost schedules, buyers submit quantities, and the market chooses price to equate reported supply and demand. In equilibrium, price is equal to each seller's marginal cost but exceeds each buyer's true willingness to pay. In each of these models, one side of the market is passive and the other side behaves strategically, anticipating the market-clearing mechanism in order to manipulate prices. As mentioned earlier, Klemperer and Meyer study markets in which sellers behave strategically in submitting their supply schedules and buyers are passive, reporting their true willingness to pay schedules. Our interest, however, is in a market where both buyers and sellers recognize their ability to unilaterally influence the price and behave strategically. In order to model this type of market, we extend the

Klemperer and Meyer model and allow buyers to behave strategically in submitting their demand schedules.

In adopting this model, however, we impose some restrictions on the schedules that the firms can report. Sellers have to submit cost schedules which come in the form $\gamma c(x/\gamma)$, and buyers have to submit valuation functions which come in the form $kv(q/k)$. In a mechanism design framework, agents can lie about their type, but they can't invent an impossible type. The admissible types in our model satisfy (1) and (2), and agents are assumed to be bound by this message space. For sellers, the message space is a one-parameter family of schedules indexed by production capacity, and for buyers, the message space is a one-parameter family of schedules indexed by consumption capacity. Therefore, a buyer's type is a capacity k, a seller's type is a capacity γ, and if the firm is vertically integrated, its type is a pair of capacities (γ, k). Agents (simultaneously) report their types to the market mechanism but, in doing so, they do not have to tell the truth. A seller can exaggerate its costs by reporting a capacity $\widehat{\gamma}$ that is less than it in fact is, and a buyer can understate its willingness to pay by reporting a capacity $\widehat{k}$ that is greater than it in fact is. Vertically integrated firms can lie about both types of capacity, but firms that have no consumption capacity or no production capacity cannot lie about this fact. In other words, the market can distinguish between firms that are only sellers or only buyers or both.

Given the agents' reports, the market mechanism chooses price p to equate reported supply and reported demand, and allocates the output efficiently. The solution is characterized by the balance equation,

$$Q = \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} x_i \tag{3}$$

and the marginal conditions,

$$v'\left(\frac{q_j}{\widehat{k}_j}\right) = p = c'\left(\frac{x_i}{\widehat{\gamma}_i}\right), i, j = 1, .., n. \tag{4}$$

From equation (4), it follows that

$$q_i = \frac{\widehat{k}_i Q}{K},\ x_i = \frac{\widehat{\gamma}_i Q}{\Gamma}, \tag{5}$$

where

$$K = \sum_{i=1}^{n} \widehat{k}_i, \ \Gamma = \sum_{i=1}^{n} \widehat{\gamma}_i. \tag{6}$$

Thus, one algorithm for computing the equilibrium price and allocation is to solve for Q using the equation

$$v'\left(\frac{Q}{K}\right) = c'\left(\frac{Q}{\Gamma}\right), \tag{7}$$

and then allocate Q to sellers and buyers using the market share equations of (5). Note that, if everyone tells the truth, then the equilibrium outcome is efficient.

This model of the market can be viewed as turning it into a black box, as in fact happens in the Cournot model, where the price formation process is not modeled explicitly. Given this black box approach, it seems appropriate to permit the market to be efficient when agents don't, in fact, exercise unilateral power. Such considerations dictate the competitive solution, given the reported types. Any other assumption would impose inefficiencies in the market mechanism, rather than having inefficiencies arise as the consequence of the rational exercise of market power by firms with significant market presence.

Each firm anticipates the market mechanism's decision rule in submitting its reports. The firms' actual types are common knowledge to the firms. Thus, in choosing their reports, firms know the true types of other firms. Let $\widehat{\gamma}$ and $\widehat{k}$ denote the vector of firm reports. Then the payoff to a vertically integrated firm i from submitting reports $(\widehat{\gamma}_i, \widehat{k}_i)$ is

$$\pi_i(\widehat{\gamma}, \widehat{k}) = k_i v\left(\frac{\widehat{k}_i Q(\widehat{\gamma}, \widehat{k})}{k_i K}\right) - \gamma_i c\left(\frac{\widehat{\gamma}_i Q(\widehat{\gamma}, \widehat{k})}{\gamma_i \Gamma}\right) - p(\widehat{\gamma}, \widehat{k}) Q(\widehat{\gamma}, \widehat{k})\left(\frac{\widehat{k}_i}{K} - \frac{\widehat{\gamma}_i}{\Gamma}\right). \tag{8}$$

If firm i is a seller with no consumption capacity, then $k_i = \widehat{k}_i \equiv 0$; similarly, if firm i is a buyer with no production capacity, then $\gamma_i = \widehat{\gamma}_i \equiv 0$.

The Nash equilibrium to the market game consists of a profile of reports $(\widehat{\gamma}, \widehat{k})$ with the property that (i) each firm correctly guesses the reports of other firms and (ii) no firm has an incentive to submit a different, feasible report. The restrictions on cost and valuations functions imply that if an equilibrium exists, it is unique.

In general, supply function models have multiple equilibria. Given the schedules submitted by its rivals and the decision rule of the market mechanism, each firm faces a fixed residual demand and has a unique profit-maximizing price-quantity pair. As a result, the firm has a lot of freedom in constructing the supply schedule at prices that are not realized in equilibrium. Klemperer and Meyer [14] reduce the multiplicity by introducing stochastic demand, which has the effect of making more of the price-quantity pairs on the supply schedule relevant to the firm's expected payoff. They show that, if the support is unbounded, then the equilibrium is unique and the equilibrium supply schedule is linear. Green and Newberry [11], Green [9][10], and Akgun [1] obtain uniqueness by restricting the supply schedules to be linear.

3 Equilibrium

We first derive and discuss equilibrium markups in vertically separated markets. We consider several special cases including the Cournot model. We then derive and discuss the equilibrium markups in vertically integrated markets.

Before stating the theorems, we require some additional notation. The market demand function Q^d is given by $v'(Q^d/K) = P$, so the market elasticity of demand is

$$\varepsilon = -\left(\frac{p}{Q}\right)\left(\frac{dQ^d}{dp}\right) = \frac{-v'(Q/K)}{(Q/K)v''(Q/K)}. \tag{9}$$

Similarly, the market supply function Q^s is given by $c'(Q^s/\Gamma) = P$, so the market elasticity of supply is

$$\eta = -\left(\frac{p}{Q}\right)\left(\frac{dQ^s}{dp}\right) = \frac{-c'(Q/\Gamma)}{(Q/\Gamma)c''(Q/\Gamma)}. \tag{10}$$

Let σ_i and s_i denote firm i's market share in production and consumption respectively. Given any profile of reports, the market shares are equal to reported capacity shares, that is,

$$\sigma_i = \frac{\widehat{\gamma}_i}{\Gamma},\ s_i = \frac{\widehat{k}_i}{K}. \tag{11}$$

Hence, in principle, reported market shares can be inferred from market shares. Finally, define

$$c_i' \equiv c'\left(\frac{\sigma_i Q}{\gamma_i}\right),\ v_i' \equiv v'\left(\frac{s_i Q}{k_i}\right). \tag{12}$$

as firm i's equilibrium marginal cost and marginal valuation.

Theorem 1 *Suppose markets are vertically separated. Then*

$$\frac{p - c_i'}{p} = \frac{\sigma_i}{\varepsilon + \eta(1 - \sigma_i)}, \tag{13}$$

and

$$\frac{v_i' - p}{p} = \frac{s_i}{\varepsilon(1 - s_i) + \eta}. \tag{14}$$

Corollary 2 *(i)* $\frac{\widehat{\gamma}}{\gamma}$ *is less than 1 and decreasing in* γ*; (ii)* $\frac{\widehat{k}}{k}$ *exceeds 1 and is increasing in* k*.*

The exercise of market power by sellers and buyers creates a double markup problem. Sellers report less than their true capacity, thereby overstating their marginal cost. Since the market mechanism equates price to reported marginal costs, it exceeds each seller's actual marginal cost. Buyers report more than their true capacity, thereby understating their true willingness to pay. As a result, price is less than each buyer's actual marginal willingness to pay. The corollary establishes that, on both sides of the market, the distortion is larger for firms with larger capacities.

As in the standard Cournot model, seller markups are constrained by the elasticity of demand. If demand is elastic, then ε is large and sellers' profit margins are small. However, the seller's margins also depend upon the elasticity of supply. In the Cournot model, if a seller restricts output, market supply falls be the same amount, and the price response depends only upon the elasticity of demand. In a supply function model like ours, if a seller tries to restrict output by reporting a higher marginal cost schedule, the reported market supply shifts to the left causing price to rise, but other sellers move up their reported supply curves, expanding their output. Thus, the fall in market supply is less than the reduction in the seller's output. The price response depends upon the slope of the reported demand curve and the slope of the reported supply curves. If marginal costs are roughly constant (i.e., $c'' \simeq 0$), then η is very large, individual sellers cannot raise price significantly by constricting their supply schedule, and the Bertrand outcome arises. On the other hand, if marginal cost curves are steeply sloped (i.e., $c''/c' \longrightarrow \infty$), then η approaches 0, and the Cournot outcome arises. Since our model treats buyers and sellers symmetrically, the same reasoning applies to buyer markups.

A particularly relevant application of our formulas is to electricity markets in which generating firms submit supply schedules, and the independent system operator (ISO) chooses price to equate reported supply to market demand. Buyers in these markets typically have inelastic demands: they simply report the amounts that they need for their retail customers to the ISO. In a Cournot model, the inelastic demands would create existence problems. However, even if we assume that demand has some elasticity, the Cournot model has difficulty explaining why markups typically increase with increases in demand; one would need to assume that the increases in demand also makes demand more inelastic. In our model, an equilibrium exists as long as *supply* is not too inelastic. Furthermore, the seller markups will increase with increases in demand even when the increases do not change the elasticity of demand. During periods of low demand, generating firms have lots of excess capacity and their markups will be low because marginal costs are fairly flat; during periods of high demand, capacity is tight and their markups will be high because generating firms are operating on the steeply rising part of their marginal cost curves.

Empirical industrial economists typically specify the first-order condition of the firm in the form

$$p = mc_i(q_i) - \lambda_i q_i \frac{\partial p}{\partial Q},$$

where λ_i is a conduct or market power parameter to be estimated. It is interpreted as firm i's conjecture on how aggregate supply will change with an increase in its output. In the Cournot model, output by rivals is fixed, so $\lambda_i = 1$. In the various empirical studies surveyed by Bresnahan [4] the empirical estimates of λ ranges from 0.05 to 0.65. More recently, Genesove and Mullin [8] report estimates of λ for the sugar industry at the turn-of-the-century ranging from 0.038 to 0.10, with the latter computed directly from the data on prices and marginal costs. Clay and Troesken [6] report similarly low estimates for the conduct parameter in the whiskey industry at the turn-of-the-century. In other words, the empirical literature finds that market power varies considerably across markets but that markets are usually much more competitive than Cournot models would predict. In our model,

$$\lambda_i = \frac{\varepsilon}{\varepsilon + (1 - \sigma_i)\eta},$$

which depends upon firm i's market share, and hence is endogenous, except in the Cournot case (i.e., $\eta = 0$). Its value is bounded between 0 and 1, with the upper bound achieved when $\eta = 0$. Since supply elasticities are typically positive and vary across industries, our model provides a potential explanation for the relatively low estimates of market power obtained in the empirical literature, and the variation in these estimates across industries.

We turn next to vertically integrated markets.

Theorem 3 *Suppose firm i is vertically integrated. Then, in any interior equilibrium, $v_i' = c_i'$ and*

$$\frac{v_i' - p}{p} = \frac{c_i' - p}{p} = \frac{s_i - \sigma_i}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}. \tag{15}$$

There are two immediate observations. First, each vertically integrated firm is technically efficient about its production; that is, its marginal cost is equal to its marginal value. Thus, the firm cannot, in the equilibrium allocation, gain from secretly producing more and consuming that output. This is not to say that the firm could not gain from the ability to secretly produce and consume, for the firm might gain from this ability by altering its reports appropriately. For example, if the firm is a net seller, it will try to raises price by restricting supply and overstating demand. It accomplishes the first by reporting a production capacity $\widehat{\gamma}$ that is less than its actual capacity γ, and the second by reporting a consumption capacity $\widehat{k}$ that is less than its actual capacity of k. A net buyer does the opposite, reporting higher production and consumption capacities than its actual capacities. Second, net buyers value the good more than the price, and net sellers value the good less than price. Thus, net buyers restrict their demand below that which would arise in perfect competition, and net sellers restrict their supply. In both cases, the gain arises because of price effects.

Theorem 4 *The (quantity weighted) average difference between marginal valuations and marginal costs satisfies:*

$$\frac{1}{p}\left(\sum_{i=1}^{n} s_i v_i' - \sum_{i=1}^{n} \sigma_i c_i'\right) = \sum_{i=1}^{n}\left(\frac{(s_i-\sigma_i)^2}{\varepsilon(1-s_i)+\sigma_i(1-\sigma_i)}\right). \tag{16}$$

Theorem 4 gives the equivalent of the Hirschman-Herfindahl Index for the present model. It has the same useful features – it depends only on market shares and elasticities. As noted above, zero net demand causes no inefficiency. Thus, an intermediate good market in which each firm is vertically integrated and supplies only itself is perfectly efficient. However, with even a small but nonzero net demand or supply, size exacerbates the inefficiency.

In this framework, the shares are of production or consumption, and not capacity. The U.S. Department of Justice Merger Guidelines [30] generally calls for evaluation shares of capacity. While our analysis begins with capacities, the shares are actual shares of production (σ_i) or consumption (s_i), rather than the capacity for production and consumption, respectively. Firms may have the same capacity in production and consumption but nevertheless choose to be a net seller or a net buyer depending upon market conditions. The use of actual consumption and production is an advantageous feature of the theory, since these values tend to be readily observed, while capacities are not. Moreover, capacity is often subject to vociferous debate by economic analysts, while the market shares may be more readily observable. Finally, the shares are shares of the total quantity and not revenue shares. However, like the Cournot model, our model is not designed to handle industries with differentiated products, which is the situation where a debate about revenue versus quantity shares arises.

4 The Constant Elasticity Case

The case where the demand and supply elasticities are constant is especially informative for studying efficiency and the effect of mergers. In particular, even when elasticities vary, formulae derived from the constant elasticity case apply approximately, with the error determined by the amount of variation in the elasticities. With constant elasticities,

$$v'(z) = z^{-\varepsilon^{-1}}, \; c'(z) = z^{\eta^{-1}}. \tag{17}$$

Let Q_f represent the first best quantity, that which arises when all firms are sincere in their behavior, and p_f be the associated price. Then

Theorem 5 *With constant elasticities, the size of the firms' misrepresentations is given by*

$$\begin{aligned} \frac{\widehat{k}_i}{k_i} &= \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{-\varepsilon} \\ \frac{\widehat{\gamma}_i}{\gamma_i} &= \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{\eta}. \end{aligned} \tag{18}$$

Moreover,

$$\frac{Q_f}{Q} = \left[\sum_{i=1}^{n} s_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{\varepsilon}\right]^{\frac{\eta}{\varepsilon+\eta}} \left[\sum_{i=1}^{n} \sigma_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{-\eta}\right]^{\frac{\varepsilon}{\varepsilon+\eta}} \tag{19}$$

and

$$\frac{p_f}{p} = \left[\frac{\sum_{i=1}^{n} \sigma_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{-\eta}}{\sum_{i=1}^{n} s_i \left(1 + \frac{s_i - \sigma_i}{\varepsilon(1-s_i) + \eta(1-\sigma_i)}\right)^{\varepsilon}}\right]^{\dfrac{1}{\varepsilon + \eta}} \tag{20}$$

Equation (17) confirms the intuition that the misrepresentation is largest for the largest net traders, and small for those not participating significantly in the intermediate good market. Indeed, the size of the misrepresentation is proportional to the discrepancy between price and marginal value or cost, as given by Theorem 1, adjusted for the demand elasticity. This is hardly surprising, since the constant demand and supply elasticities insure that marginal values can be converted to misrepresentations in a log-linear fashion.

Equation (18) provides the formula for lost trades. Here there are two effects. Net buyers under-represent their demand, but over-represent their supply. On balance, net buyers under-represent their net demands, which is why the quantity-weighted average marginal value exceeds the quantity-weighted average marginal cost. Equation (18) provides a straightforward means of calculating the extent to which a market is functioning inefficiently, both before and after a merger, at least in the case where the elasticities are approximately constant.

Equation (19) gives the effect of strategic behavior in the model on price. Note that the price can be larger, or smaller, than the efficient full-information price. Market power on the buyer's side (high values of s_i) tend to decrease the price, with buyers exercising market power. Similarly, as σ_i increases, the price tends to rise.

The constant returns to scale assumption facilitates a consideration of mergers, since the merger of two firms i and j produces a firm with consumption capacity $k_i + k_j$ and production capacity $\gamma_i + \gamma_j$, and thereby is subject to the same analysis. In what follows, we will focus on vertically separated markets. The separation of buyers and sellers in these markets means that we can study the impact of a merger in terms of shifts in the reported demand and supply schedules. The following lemma will prove useful.

Lemma 6 *In vertically separated markets with constant elasticities, capacity reports are strategic complements.*

Akgun [2004] obtains a similar result in a supply function model in which the sellers are restricted to reporting linear supply schedules and buyers have no market power. Given this restriction, he defines an equivalent game in which sellers choose the slopes of their supply schedules and shows that they are strategic complements.

Lemma 6 implies that, in a vertically separated market, a horizontal merger of sellers in vertically separated markets raises the equilibrium price, and a horizontal merger of buyers lowers the equilibrium price. The argument is as follows. Suppose two (or more) sellers merge. Holding the reports of the other sellers and buyers fixed, Corollary 2 implies that the merged firm will report a capacity that is less than the sum of the capacities reported by firms 1 and 2 when they were independent firms. It then follows from Lemma 6 that the other sellers will respond by reducing their reported capacities as well. The reported supply schedule decreases. If buyers have no market power, then the equilibrium impact of the horizontal merger is an increase in price and a reduction in output. If buyers do have market power, then Lemma 6 implies that they will respond to the reduction in Γ by decreasing their own capacity reports. Thus, the distortions on the buyer side are smaller, and the reported demand curve shifts out. The equilibrium price is higher, but the overall effect on output depends upon the market elasticities. A similar analysis applies to a horizontal merger among buyers.

Horizontal mergers are clearly profitable. The merging sellers (buyers) benefit from the merger in three ways. First, it gives them more market power. Second, the non-merging sellers (buyers) response to the merging firms' exercise of market power is not to undermine it but to magnify it by reducing their supply (demand) as well. Third, the sellers (buyers) have more bargaining power against the buyers (sellers) who have to expand their reported demand (supply).

A vertical merger between a buyer and a seller involves competing effects, which makes it difficult to sign its impact on market outcomes. Holding the reports of others sellers and buyers fixed, one can show using the first order conditions of the merging firms that the vertically integrated firm will report more production capacity and less consumption capacity. Thus, a vertical merger leads to greater efficiency, at least for the merging firms. The response of the other firms in the market will depend upon the market elasticities. The increase in reported production capacity by the merging firms will cause other sellers and buyers to increase their capacity reports, increasing their reported supply and decreasing their reported demand schedules; on the other hand, the increase in reported consumption capacity will cause other sellers and buyers to decrease their capacity reports, which has the opposite effect on their reported supply and demand schedules.

The above analysis suggests that the impact of mergers on price and output in vertically integrated markets depends critically on the elasticity parameters, and need to be computed on a case by case basis.

5 Downstream Concentration

In many, perhaps even most, applications, the assumption that a buyer in the intermediate good market can safely ignore the behavior of other firms in calculating the value of consumption is unfounded. This is particularly true when the buyers are retail firms. In this section, we extend the model to markets in which buyers are retail firms that compete in quantities in the downstream market.

The value of consumption to a retail firm is given by

$$V(q_i, k_i) = r(Q)q_i - k_i w\left(\frac{q_i}{k_i}\right). \tag{21}$$

where $r(Q)$ is the downstream inverse demand and w accounts for the selling cost. Recall that the downstream output, y_i, is equal to q_i. Firm profits are:

$$\pi_i(\gamma, k) = r(Q)q_i - k_i w\left(\frac{q_i}{k_i}\right) - \gamma_i c\left(\frac{x_i}{\gamma_i}\right) - p(q_i - x_i). \tag{22}$$

As before, we calculate the efficient solution, which satisfies:

$$p = c'(Q/\Gamma) = c'(x_i/\gamma_i) \tag{23}$$

and

$$r(Q) = p + w'(Q/K) = p + w'(q_i/k_i). \tag{24}$$

Let α be the elasticity of downstream demand, and β be the elasticity of the selling cost w. Let θ be ratio of the intermediate good price p to the final good price r. The observables of the analysis will be the market shares (both production, σ_i, and retail, s_i), the elasticity of final good demand, α, of selling cost, β, of production cost, η and the price ratio $\theta = p/r$. It will turn out that the elasticities enter in a particular way, and thus it is useful to define:

$$A = \alpha^{-1}; B = (1-\theta)\beta^{-1}; C = \theta\eta^{-1}. \tag{25}$$

We replicate the analysis of section 3 in the appendix for this more general model. The structure is to use equations (22) and (23) to construct the value to each firm of reports of

$\widehat{k}_i$ and $\widehat{\gamma}_i$. The first order conditions provide necessary conditions for a Nash equilibrium to the reporting game. These first order conditions are used to compute the price/cost margin, weighted by the firm shares. In particular, we look for a modified herfindahl index (MHI) given by:

$$MHI = \sum_{i-1}^{n} \frac{1}{r}[(r(Q) - p - w_i')s_i + (p - c_i')\sigma_i], \tag{26}$$

where $w_i = w(q_i/\gamma_i)$.

The main theorem characterizes the modified Herfindahl index for an interior solution.

Theorem 7 *In an interior equilibrium,*

$$MHI = \sum_{i=1}^{n} \left[\frac{BC(s_i - \sigma_i)^2 + ABs_i^2(1 - \sigma_i) + AC\sigma_i^2(1 - s_i)}{A(1 - s_i)(1 - \sigma_i) + B(1 - \sigma_i) + C(1 - s_i)} \right] \tag{27}$$

While complex in general, this formula has several important special cases. If $A = 0$, the downstream market has perfectly elastic demand. As a result, the r is a constant, and (26) readily reduces to (16).

When $B = 0$, there is a constant retailing cost w. This case is analogous to Cournot, in that all firms are equally efficient at selling, although the firms vary in their efficiency at producing. In this case, (26) reduces to

$$MHI|_{B=0} = \sum_{i=1}^{n} \left[\frac{AC\sigma_i^2}{A(1 - \sigma_i) + C} \right] = \sum_{i=1}^{n} \left[\frac{\theta\sigma_i^2}{\eta(1 - \sigma_i) + \theta\alpha} \right] \tag{28}$$

The Herfindahl index reflects the effect of the wholesale market through the elasticity of supply η. If $\eta = 0$, the Cournot HHI arises. For positive η, the possibility of resale increases the price/cost margin. This arises because a firm with a large capacity now has an alternative to selling that capacity on the market. A firm with a large capacity can sell some of its Cournot level of capacity to firms with a smaller capacity. The advantage of such sales to the large firm is the reduction in desire of the smaller firms to produce more, which helps increase the retail price. In essence, the larger firms buy off the smaller firms via sales in the intermediate good market, thereby reducing the incentive of the smaller firms to increase their production.

The formula (26) can be decomposed into Herfindahl-type indices for three separate markets: transactions, production and consumption. Note

$$MHI = \sum_{i=1}^{n} \left[\frac{B(1-\sigma_i)+C(1-s_i)}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \right] \left[\frac{(s_i-\sigma_i)}{C^{-1}(1-\sigma_i)+B^{-1}(1-s_i)} \right] \tag{29}$$
$$+\sum_{i=1}^{n} \left[\frac{A(1-\sigma_i)(1-s_i)}{A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)} \right] \left[\frac{Bs_i^2}{(1-s_i)} + \frac{C\sigma_i^2}{(1-\sigma_i)} \right]$$

The modified herfindahl index, MHI, is an average of three separate indicies. The first index corresponds to the transactions in the intermediate good market. In form, this term looks like the expression in Theorem 4, adjusted to express the elasticities in terms of the final output prices. The second expression is an average of the indices associated with production and consumption of the intermediate good. These two indices ignore the fact that firms consume some of their own production.

When the downstream market is very elastic, A is near zero. In this case, the MHI reduces to that of Theorem 4. This occurs because elastic demand in the downstream market eliminates downstream effects, so that the only effects arise in the intermediate good market. In contrast, when the downstream market is relatively inelastic, downstream effects dominate, and the MHI is approximately an average of the herfindahl indices for the upstream and downstream markets, viewed as separate markets.

In some sense, these limiting cases provide a resolution of the question of how to treat captive consumption. When demand is very inelastic, as with gasoline in California, then the issue of captive consumption can be ignored without major loss: it is gross production and consumption that matter. In this case, it is appropriate to view the upstream and downstream markets as separate markets and ignore the fact that the same firms may be involved in both. In particular, a merger of a pure producer and a pure retailer should raise minimal concerns. On the other hand, when demand is very elastic (A near zero), gross consumption and gross production can be safely ignored, and the market treated as if the producers and consumers of the intermediate good were separate firms, with net trades in the intermediate good the only issue that arises.[8] Few real world cases are likely to

[8]However, the denominator still depends on gross production and consumption, rather than net produc-

approximate the description of very elastic market demand.[9] However, the case of $A = 0$ also corresponds to the case where the buyers do not compete in a downstream market, and thus may have alternative applications.

In the appendix, we provide the formulae governing the special case of constant elasticities. It is straightforward to compute the reduction in quantity that arises from a concentrated market, as a proportion of the fully efficient, first-best quantity. Moreover, we provide a Mathematica 3.0 program which takes market shares as inputs and computes the capital shares of the firms, the quantity reduction and the effects of a merger.[10]

6 Application

In this section, we apply the analysis to a merger of Exxon and Mobil's gasoline refining and retailing assets in California. California's gasoline market is relatively isolated from the rest of the nation, both because of transportation costs,[11] and because of the California Air Resources Board's requirement of gasoline reformulated for lower emission, a type of gasoline known as CARB.

Available market share data is generally imperfect, because of variations due to shutdowns and measurement error, and the present analysis should be viewed as an illustration of the theory rather than a formal analysis of the Exxon-Mobil merger. Nevertheless, we have tried to use the best available data for the analysis. In Table 1, we provide a list of market shares, along with our estimates of the underlying capital shares and the post-merger market shares, which will be discussed below. The data come from Leffler and Pulliam [17].

From Table 1, it is clear that there is a significant market in the intermediate good

tion and consumption. This can matter when mergers dramatically change market shares, and even the merger of a pure producer and pure consumer can have an effect.

[9] When market demand is very elastic, it is likely that there are substitutes that have been ignored. It would usually be preferable to account for such substitutes in the market, rather than ignore them.

[10] This program is available on McAfee's website.

[11] There is currently no pipeline permitting transfer of Texas or Louisiana refined gasoline to California, and the Panama Canal can not handle large tankers, and in any case is expensive. Nevertheless, when prices are high enough, CARB gasoline has been brought from the Hess refinery in the Caribbean.

of bulk (unbranded) gasoline, prior to branding and the addition of proprietary additives. However, the actual size of the intermediate good market is larger than one might conclude from Table 1, because firms engage in swaps. Swaps trade gasoline in one region for gasoline in another. Since swaps are balanced, they will not affect the numbers in Table 1.

It is well known that the demand for gasoline is very inelastic. We consider a base case of an elasticity of demand, α, of 1/3. We estimate θ to be 0.7, an estimate derived from an average of 60.1 cents spot price for refined CARB gasoline, out of an average of 85.5 (net of taxes) at the pump.[12] We believe the selling cost to be fairly elastic, with a best estimate of $\beta = 5$. Similarly, by all accounts refining costs are quite inelastic; we use $\eta = 1/2$ as the base case. We will consider the robustness to parameters below, with $\alpha = 1/5$, $\beta = 3$, and $\eta = 1/3$.

Prior to analyzing the effect of a merger of Exxon and Mobil, we present the markup in a fully symmetric industry, given our base case assumptions. A symmetric industry involves no trade in the intermediate good. We consider variations in the number of identical firms, as a way of benchmarking the price-cost margins. These findings are presented in Table 2. Thus, a price-cost margin of 11.8 percent roughly corresponds to a symmetric industry with 9 firms, dropping to 6.8% with 15 identical firms. We generally find the gasoline industry in California to have 20% margins and approximately 95% efficiency, which is similar to an industry with 6 firms.

Table 2 also presents the calculations, for the base case elasticities, of the reduction in output for a symmetric industry. A symmetric duopoly would reduce quantity by 17.1%, which would create price increases around 75% over the competitive level, given the demand elasticity of 1/3. Even with fifteen identical firms, the quantity is reduced by 3.6%, which creates an approximately 10.8% increase in the retail price over the efficient quantity.

Table 3 presents our summary of the Exxon/Mobil merger. The first three columns provide the assumptions on elasticities that define the four rows of calculations. The fourth

[12] We will use all prices net of taxes. As a consequence, the elasticity of demand builds in the effect of taxes, so that a 10% retail price increase (before taxes) corresponds approximately to an 17% increase in the after tax price. Thus, the elasticity of 1/3 corresponds to an actual elasticity of closer to 0.2.

column provides the markups that would prevail under a fully symmetric and balanced industry, that is, one comprised of fifteen equal sized firms. This is the best outcome that can arise in the model, given the constraint of fifteen firms, and can be used a benchmark. The fifth column considers a world without refined gasoline exchange, in which all fifteen companies are balanced, and is created by averaging production and consumption shares for each firm. This calculation provides an alternative benchmark for comparison, to assess the inefficiency of the intermediate good exchange. The next four columns use the existing market shares, reported in Table 1, as an input, and then compute the price-cost margin and quantity reduction, pre-merger, post-merger, with a refinery sale, or with a sale of retail outlets, respectively.

Table 3 does not use the naive approach of combining Exxon and Mobil's market shares, an approach employed in the Department of Justice Merger Guidelines. In contrast to the merger guidelines approach, we first estimate the capital held by the firms, then combine this capital in the merger, then compute the equilibrium given the post-merger allocation of capital. The estimates are not dramatically different than those that arise using the naive approach of the merger guidelines. To model the divestiture of refining capacity, we combine only the retailing capital of Exxon and Mobil; similarly, to model the sale of retailing, we combine the refining capacity.

The estimated shares of capital are presented in Table 1. These capital shares reflect the incentives of large net sellers in the intermediate market to reduce their sales in order to increase the price, and the incentive of large net buyers to decrease their demand to reduce the price. Equilon, the firm resulting from the Shell-Texaco merger, is almost exactly balanced and thus its capital shares are relatively close to its market shares. In contrast, a net seller in the intermediate market like Chevron refines significantly less than its capital share, but retails close to its retail capital share. Arco, a net buyer of unbranded gasoline, sells less than its share from its retail stores, but refines more to its share of refinery capacity. The estimates also reflect the incentives of all parties to reduce their downstream sales to increase the price, an incentive that is larger the larger is the retailer.

The sixth column of Table 3 provides the pre-merger markup, or MHI, and is a direct calculation from equation (24) using the market shares of Table 1. The seventh, eighth and ninth columns combine Exxon and Mobil's capital assets in various ways. The seventh combines both retail and refining capital. The eigth combines retail capital, but leaves Exxon's Benicia refinery in the hands of an alternative supplier not listed in the table. This corresponds to a sale of the Exxon refinery. The ninth and last column considers the alternative of a sale of Exxon's retail outlets. (It has been announced that Exxon will sell both its refining and retailing operations in California.)

Our analysis suggests that without divestiture the merger will, under the hypotheses of the theory, have a small effect on the retail price. In the base case, the markup increases from 20% to 21%, and the retail price increases 1%.[13] Moreover, a sale of a refinery eliminates most the price increase; the predicted price increase is less than a mil. Unless retailing costs are much less elastic than we believe, a sale of retail outlets accomplishes very little. The predicted changes in prices, as a percent of the pre-tax retail price, are summarized in Table 4.

The predicted quantity, as a percentage of the fully efficient quantity, is presented in Table 3, in parentheses. The first three columns present the prevailing parameters. The next six columns correspond to the conceptual experiments discussed above. The symmetric column considers fifteen equal sized firms. The balanced asymmetric column uses the data of Table 1, but averages the refining and retail market shares to yield a no-trade initial solution. The pre-merger column corresponds to Table 1; post-merger combines Exxon and Mobil. Finally, the last two columns consider a divestiture of a refinery and retail assets, respectively. We see the effects of the merger through a small quantity reduction. Again, we see that a refinery sale eliminates nearly all of the quantity reduction.

The analysis used the computed market shares rather than the approach espoused by the U.S. Department of Justice Merger Guidelines. Our approach is completely consistent with the theory, unlike the merger guidelines approach, which sets the post-merger share

[13] The percentage increase in the retail price can be computed by noting that $p = q^{-A}$.

of the merging firms to the sum of their pre-merger shares. This is inconsistent with the theory because the merger will have an impact on all firms' shares. In Table 1, we provide our estimate of the post-merger shares along side the pre-merger shares. Exxon and Mobil were responsible for 18.6% of the refining, and we estimate that the merger will cause them to contract to 17.4%. The other firms increase their share, though not enough to offset the combined firm's contraction.

There is little to be gained by using the naive merger guidelines market shares, because the analysis is sufficiently complicated to require machine-based computation. (Such programs are simple, however, and one is provided in the Appendix.) However, we replicated the analysis using the naive market shares, and the outcomes are virtually identical. Thus, it appears that the naive approach gives the right answer in this application.

7 Conclusion

This paper presents a method for measuring industry concentration in intermediate goods markets. It is especially relevant when firms have captive consumption, that is, some of the producers of the intermediate good use some or all of their own production for downstream sales.

The major advantages of the theory are its applicability to a wide variety of industry structures, its low informational requirements, and its relatively simple formulae. The major disadvantages are the special structure assumed in the theory, and the static nature of the analysis. The special structure mirrors Cournot, and thus is subject to the same criticisms as the Cournot model. For all its defects, the Cournot model remains the standard model for antitrust analysis; the present theory extends Cournot-type analysis to a new realm.

We considered the application of the theory to the California assets of Exxon and Mobil. Several reasonable predictions emerge. First, the industry produces around 95% of the efficient quantity and the merger reduces quantity by a small amount, around 0.3%. Second, the price-cost margin is on the order of 20% and rises by a percentage point or two. Third, a sale of Exxon's refinery eliminates nearly all of the predicted price increase.

This last prediction arises because retailing costs are relatively elastic, so that firms are fairly competitive downstream. Thus, the effects of industry concentration arise primarily from refining, rather than retail. Hence the sale of a refinery (Exxon and Mobil have one refinery each) cures most of the problem associated with the merger. Fourth, the naive approach based purely on market shares gives answers similar to the more sophisticated approach of first computing the capital levels, combining the capital of the merging parties, then computing the new equilibrium market shares. Finally, it is worth noting that the computations associated with the present analysis are straightforward, and run in a few seconds on a modern PC.

As with Cournot analysis, the static nature of the theory is problematic. In some industries, entry of new capacity is sufficiently easy that entry would undercut any exercise of market power. The present theory does not accommodate entry, and thus any analysis would need a separate consideration of the feasibility and likelihood of timely entry.[14] When entry is an important consideration, the present analysis provides an upper bound on the ill-effects merger.

[14] McAfee, Simmons and Williams [21] present a Cournot-based merger evaluation theory that explicitly accommodates entry in the analysis.

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Table 1: Approximate Market Shares, California CARB Gasoline Post-Merger Numbers in Parentheses					
Company	i	Refining Market Share (σ_i)	Refining Capital Share	Retail Market Share (s_i)	Retail Capital Share
Chevron	1	26.4 (26.6)	29.5 (29.5)	19.2 (19.5)	19.0 (19.0)
Tosco	2	21.5 (21.7)	21.7 (21.7)	17.8 (18.0)	17.8 (17.8)
Equilon	3	16.6 (16.7)	16.1 (16.1)	16.0 (16.2)	16.0 (16.0)
Arco	4	13.8 (13.9)	13.0 (13.0)	20.4 (20.7)	22.0 (22.0)
Mobil	5	7.0 (13.3)	6.2 (12.4)	9.7 (17.5)	9.3 (17.8)
Exxon	6	7.0 (0.0)	6.2 (0.0)	8.9 (0.0)	8.5 (0.0)
Ultramar	7	5.4 (5.4)	4.7 (4.7)	6.8 (6.9)	6.4 (6.4)
Paramount	8	2.3 (2.3)	2.0 (2.0)	0.0 (0.0)	0.0 (0.0)
Kern	9	0.0 (0.0)	0.0 (0.0)	0.3 (0.3)	0.27 (0.27)
Koch	10	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Vitol	11	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
Tesoro	12	0.0 (0.0)	0.0 (0.0)	0.2 (0.2)	0.18 (0.18)
PetroDiamond	13	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Time	14	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)
Glencoe	15	0.0 (0.0)	0.0 (0.0)	0.1 (0.1)	0.09 (0.09)

Table 2: Markups and Quantity Reduction, in Percent, for a Symmetric Industry											
Number of Firms	2	3	4	5	6	7	8	9	10	15	20
Average Markup	74.0	42.3	30.0	22.7	18.4	15.5	13.4	11.8	10.5	6.8	5.1
Q/Q_f, in percent	82.9	87.7	92.2	94.2	95.4	96.2	96.8	97.2	97.5	98.4	98.8

Table 3: Analysis of Exxon – Mobil Merger								
Cases			Markup as a Percent of Retail Price (Quantity as a Percent of Fully Efficient Quantity in Parentheses)					
α	β	η	Symmetric	Balanced Asymmetric	Pre-merger Markup	Post-merger Markup	Refinery Sale	Retail Sale
1/3	5	½	6.9 (98.4)	18.4 (95.3)	20.0 (94.6)	21.3 (94.3)	20.1 (94.6)	21.2 (94.3)
1/5	5	½	7.9 (98.7)	21.6 (96.0)	23.6 (95.4)	25.2 (95.2)	23.7 (95.4)	25.2 (95.2)
1/3	3	½	7.0 (98.4)	18.7 (95.3)	20.3 (94.6)	21.7 (94.3)	20.5 (94.6)	21.6 (94.4)
1/3	5	1/3	8.7 (98.2)	23.0 (94.6)	25.1 (93.8)	26.7 (93.5)	25.2 (93.8)	26.7 (93.5)

Table 4: Analysis of Exxon – Mobil Merger					
Cases			Expected Percentage Quantity Decrease (Percentage Price Increase in Parentheses)		
α	β	η	Full Merger	Refinery Sale	Retail Sale
1/3	5	½	0.31 (0.94)	0.03 (0.09)	0.30 (0.90)
1/5	5	½	0.27 (1.36)	0.02 (0.11)	0.25 (1.29)
1/3	3	½	0.32 (0.97)	0.05 (0.15)	0.30 (0.89)
1/3	5	1/3	0.35 (1.06)	0.03 (0.08)	0.34 (1.03)

Appendix

Proof of Theorems 1 and 3:

Before proceeding, note that differentiating the equilibrium condition in equation (7) implies that

$$\left(\frac{K}{Q}\right)\frac{\partial Q}{\partial K}=\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}},\left(\frac{\Gamma}{Q}\right)\frac{\partial Q}{\partial \Gamma}=\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}},\ and\ \left(\frac{K}{P}\right)\frac{\partial P}{\partial K}=-\left(\frac{\Gamma}{P}\right)\frac{\partial P}{\partial \Gamma}=\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}.$$

Differentiating the firm's profit function in equation (8) and substituting the relations given above yields the following first order conditions:

$$\frac{\partial \pi_i}{\partial \hat{k}_i}=\left(v'\left(\frac{s_iQ}{k_i}\right)-p\right)\left(Q\left(\frac{1}{K}-\frac{\hat{k}_i}{K^2}\right)+s_i\frac{\partial Q}{\partial K}\right)+\left(p-c'\left(\frac{\sigma_iQ}{\gamma_i}\right)\right)\left(\sigma_i\frac{\partial Q}{\partial K}\right)-Q(s_i-\sigma_i)\frac{\partial p}{\partial K}$$

$$=\frac{Q}{K}\left[(v_i'-p)\left((1-s_i)+s_i\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+(p-c_i')\left(\sigma_i\frac{\varepsilon^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)-p(s_i-\sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right].$$

Similarly,

$$\frac{\partial \pi_i}{\partial \hat{\gamma}_i}=\left(v'\left(\frac{s_iQ}{k_i}\right)-p\right)\left(s_i\frac{\partial Q}{\partial \Gamma}\right)+\left(p-c'\left(\frac{\sigma_iQ}{\gamma_i}\right)\right)\left(Q\left(\frac{1}{\Gamma}-\frac{\hat{\gamma}_i}{\Gamma^2}\right)+\sigma_i\frac{\partial Q}{\partial \Gamma}\right)-Q(s_i-\sigma_i)\frac{\partial p}{\partial \Gamma}$$

$$=\frac{Q}{\Gamma}\left[(v_i'-p)\left(s_i\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+(p-c_i')\left(1-\sigma_i+\sigma_i\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+p(s_i-\sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right].$$

Thus,

$$\frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i}+\frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i}=v_i'-c_i'.$$

In an interior equilibrium, then, $v_i'-c_i'=0$. With either of the first order conditions, we obtain

(15).

The Case of s_i=0, σ_i>0.

If σ_i>0, the first order condition on $\hat{\gamma}_i$ holds with equality. Consequently, using s_i=0,

$$0=\frac{Q}{\Gamma}\left[\left(p-c_i'\right)\left(1-\sigma_i+\sigma_i\frac{\eta^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right)+p(0-\sigma_i)\frac{(\eta\varepsilon)^{-1}}{\varepsilon^{-1}+\eta^{-1}}\right]$$

This yields

$$\frac{p-c_i'}{p}=\frac{\sigma_i}{\varepsilon+\eta(1-\sigma_i)},$$

a formula that respects (15) (for s_i=0). In addition, we have

$$0\geq\left.\frac{\partial\pi_i}{\partial\hat{k}_i}\right|_{s_i=0},$$

which gives

$$\frac{v'(0)-p}{p}\leq\frac{-\sigma_i}{\varepsilon+\eta(1-\sigma_i)}.$$

Summarizing,

$$\frac{v_i'-p}{p}\leq\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)},\ \textit{with equality if }\ s_i>0.$$

and

$$\frac{p-c_i'}{p}\leq\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)},\ \textit{with equality if }\ \sigma_i>0.$$

Suppose k_i=0. Then $kv(q/k)=\frac{v(q/k)}{1/k}\approx qv'(q/k)\xrightarrow[k\to 0]{}0$. Thus, an agent with k_i=0 will report $\hat{k}_i=0$. Similarly, an agent with γ_i=0 produces zero. This yields (13) and (14). Q.E.D.

Proof of Theorem 4: It is readily checked that the following substitutions hold, even when a share is zero.

$$\sum_{i=1}^n v_i's_i-\sum_{i=1}^n c_i'\sigma_i=\sum_{i=1}^n\left(v_i'-p\right)s_i+\sum_{i=1}^n\left(p-c_i'\right)\sigma_i$$

$$=\sum_{i=1}^n\left(v_i'-p\right)s_i+\sum_{i=1}^n\left(p-v_i'\right)\sigma_i=\sum_{i=1}^n\left(v_i'-p\right)s_i-\sum_{i=1}^n\left(v_i'-p\right)\sigma_i$$

$$= \sum_{i=1}^{n} (v_i' - p)(s_i - \sigma_i) = \sum_{i=1}^{n} \frac{p(s_i - \sigma_i)^2}{\varepsilon(1 - s_i) + \eta(1 - \sigma_i)}. \qquad \text{Q.E.D.}$$

Proof of Corollary 2:

Suppose $\widehat{\gamma}_i > \widehat{\gamma}_j$. Then equation (5) implies that $x_i > x_j$ and hence that $\sigma_i > \sigma_j$. It then follows from the first-order conditions of firms i and j that

$$c'\left(\frac{x_i}{\gamma_i}\right) < c'\left(\frac{x_j}{\gamma_j}\right) \Longrightarrow \frac{x_i}{\gamma_i} < \frac{x_j}{\gamma_j} \Longrightarrow \frac{\widehat{\gamma}_i}{\gamma_i} < \frac{\widehat{\gamma}_j}{\gamma_j} \text{ and } \gamma_i > \gamma_j.$$

The reasoning for buyers is similar. Q.E.D.

Proof of Theorem 5: Applying (7), (15), (17):

$$\left(\frac{s_i Q}{k_i}\right)^{-\frac{1}{\varepsilon}} = v'(\frac{s_i Q}{k_i}) = p\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right) = v'\left(\frac{Q}{\sum_{i=1}^n \hat{k}_i}\right)\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right).$$

This readily gives the first part of (18); the second half is symmetric.

Rewrite (18) to obtain

$$k_i = \hat{k}_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

Thus

$$\frac{k_i}{\sum_{j=1}^n \hat{k}_j} = s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

Thus

$$\frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \hat{k}_i} = \sum_{i=1}^n s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}.$$

A similar calculation gives

$$\frac{\sum_{i=1}^n \gamma_i}{\sum_{i=1}^n \hat{\gamma}_i} = \sum_{i=1}^n \sigma_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{-\eta}.$$

Note that, with constant elasticity, actual quantity is

$$Q = \left(\sum_{i=1}^n \hat{k}_i\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\sum_{i=1}^n \hat{\gamma}_i\right)^{\frac{\varepsilon}{\varepsilon+\eta}},$$

and

$$Q_f = \left(\sum_{i=1}^n k_i\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\sum_{i=1}^n \gamma_i\right)^{\frac{\varepsilon}{\varepsilon+\eta}}.$$

Substitution gives (18). One obtains (19) from

$$\frac{p}{p_f} = \frac{v'(Q/\sum_{i=1}^n \hat{k}_i)}{v'(Q_f/\sum_{i=1}^n k_i)} = \left(\frac{Q}{Q_f}\frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \hat{k}_i}\right)^{\frac{-1}{\varepsilon}} = \left(\frac{Q_f}{Q}\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right)^{\frac{1}{\varepsilon}}$$

$$= \left[\left(\frac{\sum_{i=1}^n k_i}{\sum_{i=1}^n \hat{k}_i}\right)^{\frac{\eta}{\varepsilon+\eta}}\left(\frac{\sum_{i=1}^n \gamma_i}{\sum_{i=1}^n \hat{\gamma}_i}\right)^{\frac{\varepsilon}{\varepsilon+\eta}}\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right]^{\frac{1}{\varepsilon}} = \left[\left(\frac{\sum_{i=1}^n \gamma_i}{\sum_{i=1}^n \hat{\gamma}_i}\right)\left(\frac{\sum_{i=1}^n \hat{k}_i}{\sum_{i=1}^n k_i}\right)\right]^{\frac{1}{\varepsilon+\eta}}$$

$$= \left[\frac{\sum_{i=1}^n \sigma_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{-\eta}}{\sum_{i=1}^n s_i\left(1+\frac{s_i-\sigma_i}{\varepsilon(1-s_i)+\eta(1-\sigma_i)}\right)^{\varepsilon}}\right]^{\frac{1}{\varepsilon+\eta}}.$$

Q.E.D.

Analysis of Section 4:

Using the market calculations (23) and (24), rewrite profits to obtain

$$\pi_i = r(Q)q_i - k_i w\left(q_i/k_i\right) - \gamma_i c\left(x_i/\gamma_i\right) - p(q_i - x_i)$$

$$= (r(Q)-p)q_i - k_i w\left(q_i/k_i\right) + px_i - \gamma_i c\left(x_i/\gamma_i\right)$$

$$= w'\left(Q/K\right)q_i - k_i w\left(q_i/k_i\right) + c'\left(Q/\Gamma\right)x_i - c\left(x_i/\gamma_i\right)$$

$$= w'\left(Q/K\right)\frac{\hat{k}_i}{K}Q - k_i w\left(\frac{\hat{k}_i}{K}\frac{Q}{k_i}\right) + c'\left(Q/\Gamma\right)\frac{\hat{\gamma}_i}{\Gamma}Q - \gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q}{\gamma_i}\right).$$

The equilibrium quantity is given by

$$r(Q) - w'(Q/K) - c'(Q/\Gamma) = 0.$$

From this equation, and applying (25), it is a routine computation to show:

$$\frac{K}{Q}\frac{dQ}{dK} = \frac{K}{Q}\frac{-(w'')Q/K^2}{r' - w''/K - c''/\Gamma} = \frac{(r-p)\beta^{-1}}{r\alpha^{-1} + (r-p)\beta^{-1} + p\eta^{-1}} = \frac{B}{A+B+C}.$$

Similarly,

$$\frac{\Gamma}{Q}\frac{dQ}{d\Gamma}=\frac{C}{A+B+C}.$$

Differentiating π_i, and using the analogous notation for

$$0=\frac{K}{Q}\frac{\partial \pi_i}{\partial \hat{k}_i}=(w'-w_i')(1-s_i+s_i\frac{K}{Q}\frac{dQ}{dK})-w''\frac{s_iQ}{K}+(c'-c_i')\sigma_i\frac{K}{Q}\frac{dQ}{dK}+(w''s_i\frac{Q}{K}+c''\sigma_i\frac{Q}{\Gamma})\frac{K}{Q}\frac{dQ}{dK}$$

$$=(w'-w_i')(1-s_i+s_i\frac{K}{Q}\frac{dQ}{dK})+(c'-c_i')\sigma_i\frac{K}{Q}\frac{dQ}{dK}-\beta^{-1}w's_i+(\beta^{-1}w's_i+\eta^{-1}c'\sigma_i)\frac{K}{Q}\frac{dQ}{dK}$$

$$=(w'-w_i')(1-s_i+s_i\frac{K}{Q}\frac{dQ}{dK})+(c'-c_i')\sigma_i\frac{K}{Q}\frac{dQ}{dK}-r\left(Bs_i-(Bs_i+C\sigma_i)\frac{K}{Q}\frac{dQ}{dK}\right)$$

$$=(w'-w_i')(1-s_i+s_i\frac{K}{Q}\frac{dQ}{dK})+(c'-c_i')\sigma_i\frac{K}{Q}\frac{dQ}{dK}-r\left(Bs_i\left(1-\frac{K}{Q}\frac{dQ}{dK}\right)-C\sigma_i\frac{K}{Q}\frac{dQ}{dK}\right)$$

Similarly, and symmetrically,

$$0=\frac{\Gamma}{Q}\frac{\partial \pi_i}{\partial \hat{\gamma}_i}=(w'-w_i')s_i\frac{\Gamma}{Q}\frac{dQ}{d\Gamma}+(c'-c_i')(1-\sigma_i+\sigma_i\frac{\Gamma}{Q}\frac{dQ}{d\Gamma})-r\left(C\sigma_i\left(1-\frac{\Gamma}{Q}\frac{dQ}{d\Gamma}\right)-Bs_i\frac{\Gamma}{Q}\frac{dQ}{d\Gamma}\right).$$

These equations can be expressed, substituting the elasticities with respect to capacity, as

$$\begin{bmatrix} A+B+C-s_i(A+C) & B\sigma_i \\ Cs_i & A+B+C-\sigma_i(A+B)\end{bmatrix}\begin{pmatrix}(w'-w_i')/r \\ (c'-c_i')/r\end{pmatrix}=\begin{pmatrix}Bs_i(A+C)-BC\sigma_i \\ c\sigma_i(A+B)-BCs_i\end{pmatrix}$$

The determinant of the left hand side matrix is given by

$$DET=[A+B+C-s_i(A+C)][A+B+C-\sigma_i(A+B)]-BCs_i\sigma_i$$

$$=(A+B+C)[(A+B+C)(1-s_i)(1-\sigma_i)+Bs_i(1-\sigma_i)+C(1-s_i)\sigma_i]$$

$$=(A+B+C)[A(1-s_i)(1-\sigma_i)+B(1-\sigma_i)+C(1-s_i)].$$

Thus,

$$\begin{pmatrix}(w' - w_i')/r \\ (c' - c_i')/r\end{pmatrix} = \frac{1}{DET}\begin{bmatrix} A+B+C-\sigma_i(A+B) & -B\sigma_i \\ -Cs_i & A+B+C-s_i(A+C)\end{bmatrix}\begin{pmatrix} Bs_i(A+C) - BC\sigma_i \\ c\sigma_i(A+B) - BCs_i\end{pmatrix}$$

$$= \frac{1}{DET}\begin{pmatrix}(A+B+C-\sigma_i(A+B))(Bs_i(A+C)-BC\sigma_i) - B\sigma_i(c\sigma_i(A+B)-BCs_i) \\ -Cs_i(Bs_i(A+C)-BC\sigma_i) + (A+B+C-s_i(A+C))(c\sigma_i(A+B)-BCs_i)\end{pmatrix}$$

$$= \frac{A+B+C}{DET}\begin{pmatrix}(Bs_i(A+C)-BC\sigma_i) - ABs_i\sigma_i \\ (C\sigma_i(A+B)-BCs_i) - ACs_i\sigma_i\end{pmatrix} = \frac{A+B+C}{DET}\begin{pmatrix} B[C(s_i-\sigma_i) + As_i(1-\sigma_i)] \\ C[B(\sigma_i - s_i) + A\sigma_i(1-s_i)]\end{pmatrix}.$$

Thus,

$$MHI = \sum_{i=1}^{n}\left(\frac{(r(Q) - p - w_i')s_i + (p - c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{(w' - w_i')s_i + (c' - c_i')\sigma_i}{r}\right)$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left(s_i B[C(s_i-\sigma_i) + As_i(1-\sigma_i)] + \sigma_i C[B(\sigma_i - s_i) + A\sigma_i(1-s_i)]\right)\right]$$

$$= \sum_{i=1}^{n}\left[\left(\frac{A+B+C}{DET}\right)\left[BC(s_i-\sigma_i)^2 + ABs_i^2(1-\sigma_i) + AC\sigma_i^2(1-s_i)\right]\right]$$

$$= \sum_{i=1}^{n}\frac{BC(s_i-\sigma_i)^2 + ABs_i^2(1-\sigma_i) + AC\sigma_i^2(1-s_i)}{A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)}.$$

Q.E.D.

Derivation of the optimal quantity formula with constant elasticities

$$\begin{pmatrix} w' - w_i' \\ c' - c_i'\end{pmatrix} = r\begin{pmatrix}\dfrac{B[C(s_i-\sigma_i) + As_i(1-\sigma_i)]}{A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)} \\ \dfrac{C[B(\sigma_i - s_i) + A\sigma_i(1-s_i)]}{A(1-s_i)(1-\sigma_i) + B(1-\sigma_i) + C(1-s_i)}\end{pmatrix} \equiv r\begin{pmatrix}\psi_i \\ \chi_i\end{pmatrix}.$$

Then

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{\beta}} = w_i' = w' - r\psi_i = r(1-\theta-\psi_i)$$

or,

$$k_i = s_i Q[r(1-\theta-\psi_i)]^{-\beta}$$

Similarly,

$$\gamma_i = \sigma_i Q[r(\theta-\chi_i)]^{-\eta}$$

The efficient solution satisfies $r - w'\left(Q_f \big/ \sum k_i\right) - c'\left(Q_f \big/ \sum \gamma_i\right) = 0$, or

$$0 = Q_f^{-\frac{1}{\alpha}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\beta}} r\left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\eta}} r\left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}$$

Thus, substituting and dividing by $r=Q^{-1/\alpha}$

$$0 = \left(\frac{Q_f}{Q}\right)^{\frac{-1}{\alpha}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\beta}}\left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} - \left(\frac{Q_f}{Q}\right)^{\frac{1}{\eta}}\left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}$$

or

$$1 = \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{\beta}}\left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} + \left(\frac{Q_f}{Q}\right)^{A+\frac{1}{\eta}}\left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}.$$

or

$$1 = \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{\beta}\right)}\left(\sum_{i=1}^{n} s_i[1-\theta-\psi_i]^{-\beta}\right)^{\frac{-1}{\beta}} + \left(\frac{Q}{Q_f}\right)^{-\left(A+\frac{1}{\eta}\right)}\left(\sum_{i=1}^{n} \sigma_i[\theta-\chi_i]^{-\eta}\right)^{\frac{-1}{\eta}}.$$

This equation can be solved for the ratio $Q_f \big/ Q$ which yields the underproduction.

Analysis with a Competitive Fringe

In many circumstances, there are firms that are best modeled as price-takers. Moreover, in some instances, the competitive fringe may use a distinct production technology, and therefore have different cost elasticities. This section replicates the general model with a competitive fringe. Variables associated with the fringe are denoted with the subscript 0. For example, q_0 is the downstream quantity of the fringe, and β_0 the fringe's retailing cost elasticity. We will only consider the constant elasticity case here, although the more general model follows directly.

The efficient solution satisfies:

$$0 = Q^{-1/\alpha} - \left(\frac{Q-q_0}{K}\right)^{1/\beta} - \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta},$$

$$\left(\frac{q_0}{k_0}\right)^{\frac{1}{\beta_0}} = \left(\frac{Q-q_0}{K}\right)^{1/\beta} \text{ and } \left(\frac{x_0}{\gamma_0}\right)^{\frac{1}{\eta_0}} = \left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta}$$

Thus

$$\left(\frac{Q-q_0}{K}\right) = \left(\frac{q_0}{k_0}\right)^{\frac{\beta}{\beta}}$$

Therefore,

$$0 = \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\left(\frac{q_0}{k_0}\right)^{\beta/\beta_0 - 1}\frac{1}{k_0}\right)$$

$$= \frac{dQ}{K} - \frac{Q-q_0}{K^2}dK - dq_0\left(\frac{1}{K} + \frac{\beta}{\beta_0}\frac{1}{q_0}\frac{Q-q_0}{K}\right)$$

$$dq_0 = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{Q-q_0}{q_0}} = \frac{dQ - \frac{Q-q_0}{K}dK}{1 + \frac{\beta}{\beta_0}\frac{1-s_0}{s_0}}.$$

Similarly,

$$dx_0 = \frac{dQ - \frac{Q-x_0}{\Gamma} d\Gamma}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}}.$$

Differentiating the price=marginal cost equation,

$$0 = dQ\left[-\frac{1}{\alpha} Q^{-1/\alpha - 1} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{1}{K} - \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta - 1} \frac{1}{\Gamma} \right]$$

$$+ \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{dq_0}{K} + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta - 1} \frac{Q-q_0}{K^2} dK + \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta - 1} \frac{dx_0}{\Gamma}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{Q}{Q-q_0} - \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{Q}{Q-x_0} \right]$$

$$+ \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{1}{Q-q_0} \frac{dQ - \frac{Q-q_0}{K} dK}{1 + \frac{\beta}{\beta_0} \frac{1-s_0}{s_0}} + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K}$$

$$+ \frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{1}{Q-x_0} \frac{dQ}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}}$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{Q}{Q-q_0} \left(1 - \frac{1}{1 + \frac{\beta(1-s_0)}{\beta_0 s_0}}\right)\right]$$

$$+ \frac{dQ}{Q}\left[-\frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{Q}{Q-x_0} \left(1 - \frac{1}{1 + \frac{\eta}{\eta_0} \frac{1-\sigma_0}{\sigma_0}}\right)\right] + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K}\left(1 - \frac{1}{1 + \frac{\beta}{\beta_0} \frac{1-s_0}{s_0}}\right)$$

$$= \frac{dQ}{Q}\left[-\frac{1}{\alpha} Q^{-1/\alpha} - \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{1}{1-s_0}\left(\frac{\beta(1-s_0)}{\beta_0 s_0 + \beta(1-s_0)}\right)\right]$$

$$+ \frac{dQ}{Q}\left[-\frac{1}{\eta}\left(\frac{Q-x_0}{\Gamma}\right)^{1/\eta} \frac{1}{1-s_0}\left(\frac{\eta(1-\sigma)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\right)\right] + \frac{1}{\beta}\left(\frac{Q-q_0}{K}\right)^{1/\beta} \frac{dK}{K}\left(\frac{\beta(1-s_0)}{\beta_0 s_0 + \beta(1-s_0)}\right)$$

$$= -\frac{dQ}{Q}\left[Ar + \frac{(1-\theta)r}{\beta(1-s_0)+\beta_0 s_0} + \frac{\theta r}{\eta(1-\sigma_0)+\eta_0\sigma_0}\right] + (1-s_0)\frac{dK}{K}\frac{(1-\theta)r}{\beta(1-s_0)+\beta_0 s_0}$$

Define

$$A = \frac{1}{\alpha}, \quad B = \frac{1-\theta}{\beta(1-s_0)+\beta_0 s_0}, \quad C = \frac{\theta}{\eta(1-\sigma_0)+\eta_0\sigma_0}$$

Then,

$$\frac{K}{Q}\frac{dQ}{dK} = (1-s_0)\frac{B}{A+B+C}.$$

$$\frac{d}{dK}\frac{Q-q_0}{K} = \frac{dQ-dq_0}{K} - \frac{Q-q_0}{K^2}dK = \frac{1}{K}\left[dQ - \frac{dQ - \frac{Q-q_0}{K}dK}{1+\frac{\beta}{\beta_0}\frac{1-s_0}{s_0}} - \frac{Q-q_0}{K}dK \right]$$

$$= \frac{1}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(dQ - \frac{Q-q_0}{K}dK\right) = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{dQ}{dK} - \frac{Q-q_0}{K}\right)$$

$$= \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{K}{Q}\frac{dQ}{dK} - \frac{Q-q_0}{Q}\right)\frac{Q}{K} = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{(1-s_0)B}{A+B+C} - (1-s_0)\right)\frac{Q}{K}$$

$$= \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{B}{A+B+C} - 1\right)\frac{Q-q_0}{K} = \frac{dK}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}$$

$$\frac{d}{dK}\frac{Q-x_0}{\Gamma} = \frac{1}{\Gamma}\left[\frac{dQ}{dK} - \frac{dQ/dK}{1+\frac{\eta}{\eta_0}\frac{1-\sigma_0}{\sigma_0}}\right] = \frac{1}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

The firm's profits are

$$\pi = r(Q)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{(Q-q_0)}{k_i}\right) - \gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q-x_0}{\gamma_i}\right) + p\left(\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0) - \frac{\hat{k}_i}{K}(Q-q_0)\right)$$

$$= w'\left(\frac{Q-q_0}{K}\right)\frac{\hat{k}_i}{K}(Q-q_0) - k_i w\left(\frac{\hat{k}_i}{K}\frac{Q-q_0}{k_i}\right) + c'\left(\frac{Q-x_0}{\Gamma}\right)\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0) - \gamma_i c\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{Q-x_0}{\gamma_i}\right).$$

$$\frac{\partial \pi}{\partial \hat{k}_i} = (w' - w_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{k}_i(Q-q_0)}{K} + w''\left(\frac{d}{dK}\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$
$$+ (c' - c_i')\frac{\partial}{\partial \hat{k}_i}\frac{\hat{\gamma}_i(Q-x_0)}{\Gamma} + \left(c''\right)\left(\frac{\hat{\gamma}_i}{\Gamma}\frac{d}{dK}(Q-x_0)\right)\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\left(\frac{(Q-q_0)}{K} - \frac{\hat{k}_i}{K}\left(\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\right)$$

$$+ w''\left(\frac{1}{K}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{Q-q_0}{K}\right)\frac{\hat{k}_i(Q-q_0)}{K}$$

$$+ (c' - c_i')\frac{\hat{\gamma}_i}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ c''\frac{\hat{\gamma}_i}{\Gamma}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})\frac{\hat{\gamma}_i}{\Gamma}(Q-x_0)$$

$$= (w' - w_i')\frac{(Q-q_0)}{K}\left(1 - \frac{s_i}{1-s_0}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right)$$

$$+ \frac{w'}{\beta}\left[\frac{\beta(1-s_0)}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\frac{s_i}{1-s_0}\frac{Q-q_0}{K}$$

$$+ (c' - c_i')\frac{\sigma_i}{1-\sigma_0}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C}$$

$$+ \frac{c'}{\eta}\frac{\sigma_i}{1-\sigma_0}\frac{\eta(1-\sigma_0)}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{Q-q_0}{K}\frac{B}{A+B+C})$$

Thus,

$$0 = (w' - w_i')\left(1 - \left[\frac{\beta s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right) + w'\left[\frac{s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+ (c' - c_i')\frac{\eta\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C} + c'\frac{\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}$$

Thus,

$$0 = \frac{(w' - w_i')}{r}\left(1 - \left[\frac{\beta s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)\right) + (1-\theta)\left[\frac{s_i}{\beta(1-s_0)+\beta_0 s_0}\right]\left(\frac{A+C}{A+B+C}\right)$$

$$+ \frac{c' - c'}{r}{}_i\,\frac{\eta\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C} + \theta\frac{\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0}\frac{B}{A+B+C}$$

Employing the definitions $\psi_i = \dfrac{w' - w_i'}{r}$, $\chi_i = \dfrac{c' - c_i'}{r}$ and using the analogous equation from the optimization of $\hat{\gamma}_i$ we have

$$\begin{bmatrix} A+B+C - \dfrac{\beta(A+C)s_i}{\beta(1-s_0)+\beta_0 s_0} & \dfrac{\eta B\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \\ \dfrac{\beta C s_i}{\beta(1-s_0)+\beta_0 s_0} & A+B+C - \dfrac{\eta(A+B)\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \end{bmatrix}\begin{pmatrix}\psi_i \\ \chi_i\end{pmatrix} = \begin{pmatrix} B(A+C)s_i - BC\sigma_i \\ C(A+B)\sigma_i - BCs_i \end{pmatrix}$$

Let $\overline{\beta} = \beta(1-s_0)+\beta_0 s_0$ and $\overline{\eta} = \eta(1-\sigma_0)+\eta_0\sigma_0$. Then the determinant of the LHS matrix is

$$DET = (A+B+C)^2 - (A+B+C)\left[\frac{\beta}{\overline{\beta}}(A+C)s_i + \frac{\eta}{\overline{\eta}}(A+B)\sigma_i\right] + \frac{\beta}{\overline{\beta}}\frac{\eta}{\overline{\eta}}\left[(A+C)(A+B) - BC\right]s_i\sigma_i$$

$$= (A+B+C)\left[(A+B+C) - \left[\frac{\beta}{\overline{\beta}}(A+C)s_i + \frac{\eta}{\overline{\eta}}(A+B)\sigma_i\right] + \frac{\beta}{\overline{\beta}}\frac{\eta}{\overline{\eta}}As_i\sigma_i\right]$$

$$= (A+B+C)\left[A\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right) + B\left(1-\frac{\eta}{\overline{\eta}}\sigma_i\right) + C\left(1-\frac{\beta}{\overline{\beta}}s_i\right)\right]$$

Thus

$$\begin{pmatrix} \psi_i \\ \chi_i \end{pmatrix} = \frac{1}{DET} \begin{bmatrix} A+B+C-\dfrac{\eta(A+B)\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} & \dfrac{-\eta B\sigma_i}{\eta(1-\sigma_0)+\eta_0\sigma_0} \\ \dfrac{-\beta C s_i}{\beta(1-s_0)+\beta_0 s_0} & A+B+C-\dfrac{\beta(A+C)s_i}{\beta(1-s_0)+\beta_0 s_0} \end{bmatrix} \begin{pmatrix} B(A+C)s_i - BC\sigma_i \\ C(A+B)\sigma_i - BCs_i \end{pmatrix}$$

Thus,

$$\psi_i = \frac{B}{DET}\left[\left(A+B+C-\frac{\eta}{\bar{\eta}}(A+B)\sigma_i\right)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}\sigma_i\left(C(A+B)\sigma_i - BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}(A+B)\sigma_i\left((A+C)s_i\right) - \frac{\eta}{\bar{\eta}}\sigma_i\left(-BCs_i\right)\right]$$

$$= \frac{B}{DET}\left[(A+B+C)\left((A+C)s_i - C\sigma_i\right) - \frac{\eta}{\bar{\eta}}(A+B+C)As_i\sigma_i\right]$$

$$= \frac{B(A+B+C)}{DET}\left[\left((A+C)s_i - C\sigma_i\right) - A\frac{\eta}{\bar{\eta}}s_i\sigma_i\right] = \frac{B(A+B+C)}{DET}\left[C(s_i-\sigma_i) + As_i\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right)\right]$$

$$= \frac{B\left[C(s_i-\sigma_i) + As_i\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right)\right]}{A\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + B\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + C\left(1-\frac{\beta}{\bar{\beta}}s_i\right)}$$

Analogously,

$$\chi_i = \frac{C\left[B(\sigma_i - s_i) + A\sigma_i\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\right]}{A\left(1-\frac{\beta}{\bar{\beta}}s_i\right)\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + B\left(1-\frac{\eta}{\bar{\eta}}\sigma_i\right) + C\left(1-\frac{\beta}{\bar{\beta}}s_i\right)}$$

Computations are performed as follows. First, note that

$$\left(\frac{s_i Q}{k_i}\right)^{\frac{1}{\beta}} = w_i' = w' - r\psi_i = r(1-\theta-\psi_i)$$

or,

$$k_i = s_i Q[r(1-\theta-\psi_i)]^{-\beta}$$

Similarly,

$$\gamma_i = \sigma_i Q[r(\theta-\chi_i)]^{-\eta}$$

Choose price and quantity units so that the initial price and quantity are both unity. Then capacities are readily computed from these equations, and are correct up to a scalar. The price is generally $r = Q^{-A}$, and thus the effect of a change in the allocation of the capacities can be computed by solving for the market shares the equations

$$F = k_i^{1/\beta}(1-\theta-\psi_i) - s_i^{1/\beta} Q^{1/\beta + A} = 0$$

and $G = \gamma_i^{1/\eta}(\theta-\chi_i) - \sigma_i^{1/\eta} Q^{1/\eta + A} = 0$

The strategy for computation is to guess values of Q and θ, then see if the firms' desired shares, as given by the solutions to F=G=0, sum to one. F and G have the useful properties that they are both positive at s_i=σ_i=0, F is increasing in σ_i and decreasing in s_i, and G is decreasing in σ_i and increasing in s_i. Thus, a search starting at (0,0) and always increasing in both arguments finds a solution for the desired shares although the solution may be the upper bound of 1. These solutions provide a sum of shares; the two equations

(*) $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} \sigma_i = 1$

are then used to compute the solution for Q and θ, the remaining unknowns. The unknowns are bounded, θ in [0,1] and Q greater than zero and no greater than the efficient quantity. The latter is calculable directly from the capacities. The desired shares are decreasing in the total quantity Q, guaranteeing a unique solution in Q for any given θ. Generally, as θ rises, the desired values of σ_i rise and s_i fall, but we don't have a proof for this. Mathematica 3.0 invariably finds a solution while Mathematica 2.2. did not.